Answeres to calculus sample problems III for final test in Fall Semester

1. Let P be any partition of $[a,b]$. Then $m_i = 0$, $M_i = 2$ for every index $i$, thus S(P)=0 and T(P)= $2(b-a)$. Hence $\underline{\int_a^b} f$ =0 and $\overline{\int_a^b} f = 2(b-a)$.

2. Since $\int_x^{x^2} \underline{\sin^6 t}\, dt = \int_x^0 \underline{\sin^6 t}\, dt + \int_0^{x^2} \underline{\sin^6 t}\, dt = -\int_0^x \underline{\sin^6 t}\, dt + \int_0^{x^2} \underline{\sin^6 t}\, dt$, we have $f'(x) = -(\sin^6 x) + (2x)\sin^6 x^2$

3. $\int \frac{(\sqrt{x}+1)^2}{\sqrt{x}} dx = \int 2u^2 du = \frac{2}{3}(\sqrt{x}+1)^3$ +c

4. Let $P_n$ be a partition of $[1,2]$ into n subintervals of equal length. Then $S(P_n)$=1, $T(P_n)=1+\frac{1}{n}$, and $1 \le \underline{\int_1^2} f \le \overline{\int_1^2} f \le 1+\frac{1}{n}$. Let $n \to \infty$, it follows that $1 \le \underline{\int_1^2} f \le \overline{\int_1^2} f \le 1$, thus $\underline{\int_1^2} f = \overline{\int_1^2} f$ =1.

5. $\int_{-2}^{2}(|x-1|+|x+1|)dx = \int_{-2}^{-1}(|x-1|+|x+1|)dx + \int_{-1}^{1}(|x-1|+|x+1|)dx +$
$\int_{1}^{2}(|x-1|+|x+1|)dx = \int_{-2}^{-1}(-(x-1)-(x+1))dx + \int_{-1}^{1}(-(x-1)+(x+1))dx +$
$\int_{1}^{2}((x-1)+(x+1))dx$ =3+4+3=10

6. $\int (\sin^2 \underline{2x})dx = \int \frac{1-\cos 4x}{2} dx = \frac{x}{2} - \frac{1}{8}\sin 4x$ +c

7. $\int_1^3 f(x)dx = \frac{32}{3} = f(c)(3-1) = 2(c^2+1)$. Thus, $c = \sqrt{\frac{13}{3}}$

8. (i) Since $\sum_{i=1}^{n}(i+1)^3 = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$, we have $\sum_{i=1}^{n} i^2 =$
$\frac{1}{3}\left(\sum_{i=1}^{n}(i+1)^3 - \sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) = \frac{1}{3}\left((n+1)^3 - 1 - \frac{3n(n+1)}{2} - n\right) =$
$\frac{n(n+1)(2n+1)}{6}$ (ii) $\int_0^1 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{i}{n})^2 \frac{1}{n} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$

9. The equation of the line passing through $(4,2),(1,-1)$ is $x = y+2$. Thus ,the area is $\int_{-1}^{2}(y+2-y^2)dy = \frac{9}{2}$

10. $8\int_0^r (\sqrt{r^2-x^2})^2 dx = 8\int_0^r (r^2-x^2)dx = \frac{16r^3}{3}$

11. $V = 2\int_2^4 2\pi x\sqrt{1-(x-3)^2}\,dx = 4\pi\int_{-1}^{1}(u+3)\sqrt{1-u^2}\,du = 4\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(\sin t+3)\cos^2 t\,dt =$

$$4\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 3\cos^2 t dt = 6\pi^2 .$$

12. Let $x = \cos^3 t$ , $y = \sin^3 t$ . Then

$$s = 4\int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = 4\int_0^{\frac{\pi}{2}} 3\cos t \sin t dt = 4\int_0^{\frac{\pi}{2}} \frac{3}{2}\sin 2t dt = 6$$

13. Let $f(x) = \ln x$ , then $f'(1) = 1$ which means $\lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} = 1$ or

$\lim_{h\to 0}\frac{\ln(1+h)}{h} = 1$ . This implies $\lim_{h\to 0}\exp\frac{\ln(1+h)}{h} = e \Rightarrow$

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e \Rightarrow \lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

14. $\frac{d}{dx}(\sin x)^{\sin x} = u^u(1+\ln u)\cos x = (\sin x)^{\sin x}(1+\ln\sin x)\cos x$

15. $\int_0^{\frac{\pi}{4}} \underline{\tan x}dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx = -\ln|\cos x|\Big|_0^{\frac{\pi}{4}} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2}$