Answers to calculus sample problems II for midterm test in Fall Semester

1. Choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$, then we have $\delta \le 1$, and $\delta \le \frac{\varepsilon}{5}$. If $0 < |x-2| < \delta$ then $|x+2| < 5$ and $|x-2| < \frac{\varepsilon}{5}$. Hence, $|x^2-4| = |x+2||x-2| < \varepsilon$.

2. $$\lim_{x\to1}\frac{\sqrt{3+\sqrt[3]{x}}-2}{x-1} = \lim_{x\to1}\frac{\sqrt[3]{x}-1}{(x-1)(\sqrt{3+\sqrt[3]{x}}+2)} =$$

$$\lim_{x\to1}\frac{x-1}{(x-1)(\sqrt{3+\sqrt[3]{x}}+2)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)} = \lim_{x\to1}\frac{1}{(\sqrt{3+\sqrt[3]{x}}+2)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)} = \frac{1}{12}$$

3. $$\lim_{x\to\infty}\sqrt[4]{x^4+x^3}-x = \lim_{x\to\infty}\frac{x^4+x^3-x^4}{(x^4+x^3)^{\frac{3}{4}}+(x^4+x^3)^{\frac{2}{4}}x+(x^4+x^3)^{\frac{1}{4}}x^2+x^3} =$$

$$\lim_{x\to\infty}\frac{1}{(1+x^{-1})^{\frac{3}{4}}+(1+x^{-1})^{\frac{2}{4}}+(1+x^{-1})^{\frac{1}{4}}+1} = \frac{1}{4}$$

4. $$\lim_{x\to0}f(x) = f(0) \Rightarrow \lim_{x\to0}(x\sin\frac{1}{x}) = c \Rightarrow 0 = c$$

5. Yes. $$f'(0) = \lim_{x\to0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0}\frac{x^2\sin\frac{1}{x}-0}{x-0}\ \lim_{x\to0}\frac{x^2\sin\frac{1}{x}}{x} = \lim_{x\to0}x\sin\frac{1}{x} = 0$$

6. $(fg)'(x) = (2x^3)' = 6x^2$, $f'(g(x)) = 2(2x) = 4x$, $(f\circ g)'(x) = f'(g(x))g'(x) = 8x$, and $(g\circ f)'(x) = (2x^2)' = 4x$

7. $$f(x) = \frac{x(\sin x^2)}{2x+1} \Rightarrow f'(x) = \frac{(2x+1)(x\sin x^2)'-(2x+1)'(x\sin x^2)}{(2x+1)^2} =$$

$$\frac{(2x+1)(x(\cos x^2)\cdot 2x+\sin x^2)-2(x\sin x^2)}{(2x+1)^2}$$

8. $$D\sin(\cos(\sin x)) = \cos(\cos(\sin x))D\cos(\sin x) = -\underline{\cos(\cos(\sin x))}\underline{\sin(\sin x)}D\sin x =$$

$$-\underline{\cos(\cos(\sin x))}\underline{\sin(\sin x)}\cos x$$

9. $$x+y+\sin(xy^2) = 1 \Rightarrow 1+y'+(y^2+2xyy')\cos(xy^2) = 0 \Rightarrow y' = -\frac{1+y^2\cos(xy^2)}{1+2xy\cos(xy^2)}$$

10. $$x^2+xy+y^2 = 3 \Rightarrow 2x+(y+xy')+2yy' = 0 \Rightarrow y' = -\frac{2x+y}{x+2y} \Rightarrow y' = -\frac{2+1}{1+2} = -1$$

at the point $(1,1)$. Hence the equation of the tangent line passing through $(1,1)$ is $y-1 = -(x-1)$.

11. Let $f(z) = \sin z$. By mean value, we have $\dfrac{f(x) - f(y)}{x - y} = f'(z^*)$ for some z^* between x and y. This implies $\dfrac{|\sin x - \sin y|}{|x - y|} = \cos z^* \le 1$, i.e.

$|\sin x - \sin y| \le |x - y|$.

12. $A = \pi r^2 \Rightarrow \dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$

13. omitted

14. Since $\sqrt{(x-4)^2 + 9} < \sqrt{(-x-4)^2 + 9}$ for $x > 0$, we need only consider the case $x > 0$. Solve the equation $\left(\dfrac{x}{\sqrt{(x-4)^2 + 9}}\right)' = 0$, we obtain $x = \dfrac{25}{4}$, and $f(\dfrac{25}{4}) = \dfrac{5}{3}$, which is the maximum. (Note that $f(0) = 0$, and $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{(x-4)^2 + 9}} = 1$.)

15. Let the altitude of the cylinder be $2x$. It follows that the volume of the cylinder is $V = (\sqrt{1 - x^2})^2 \pi (2x)$. Solve the equation $\dfrac{dV}{dx} = 0$, we have $x = \dfrac{1}{\sqrt{3}}$. Hence the maximum volume is $V = \dfrac{2}{3}\pi(\dfrac{2}{\sqrt{3}}) = \dfrac{4\sqrt{3}\pi}{9}$.